Background data for the paper “Large-scale lithospheric stress field and topography induced by global mantle circulation” by Steinberger, Schmeling and Marquart

1. Further details of the calculation of stresses within the lithosphere due to forces acting at its base

1.1 An estimate of the relative importance of flexural stresses

In the absence of flexural rigidity and with no water cover, the stress anomaly τ_i associated with a vertical lithospheric deflection h_d is of the order

$$\tau_i \approx \rho \cdot g \cdot h_d$$

where g is gravity and ρ is the density of the lithosphere. Using thin-plate theory, it follows from Hooke's law a relation

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx}$$

between flexural stresses σ_{xx} and strains ε_{xx}, where E is Young's modulus and ν is Poisson's ratio. At the top and bottom of the plate of thickness t_L, the strains associated with lithospheric flexure reach a maximum value

$$\varepsilon_{xx,max} = \frac{t_L}{2} \cdot \frac{4\pi^2}{\lambda^2} \cdot h_d$$

for a wavelength λ. Therefore

$$\frac{\sigma_{xx,max}}{\tau_i} = \frac{E}{1-\nu^2} \cdot \frac{t_L}{2} \cdot \frac{4\pi^2}{\lambda^2} \cdot \frac{1}{\rho g}$$

i.e. the relative magnitude of flexural stresses gets smaller for longer wavelengths. For $E = 7 \cdot 10^{10}$ Pa, $\nu = 0.25$, $\rho = 3300\mathrm{kg/m}^3$, g=10m/s^2 and the shortest wavelengths considered λ=1250 km (corresponding to spherical harmonic degree 32) it follows $\sigma_{xx,max}/\tau_i =$

0.71, i.e. for the shortest wavelengths considered, both kinds of stresses reach similar magnitudes.

1.2 Derivation and interpretation of Eqs. 4 and 5 of the main text

The following idealized problem is solved here: A viscous (case 1) or elastic (case 2) spherical layer (called the "lithosphere") of thickness t_L has stresses τ_{rr}, $\tau_{r\vartheta}$ and $\tau_{r\varphi}$ acting at its base. The horizontal displacement rates (case 1) or displacements (case 2) of the layer are computed. The following simplifying assumptions are made:

1. The thickness of the layer t_L is small compared to the length scale over which stresses change, hence also small compared to r_E.
2. The upper surface is stress free
3. Vertical variations of $\tau_{\vartheta\vartheta}$, $\tau_{\vartheta\varphi}$ and $\tau_{\varphi\varphi}$ are neglected (because flexural stresses are neglected, as explained in point 1.1)
4. Stresses τ_{rr} acting at the base of the lithosphere cause vertical displacements $h_d = \tau_{rr}/(\Delta\rho g) + h_{d,0}$ (because flexural rigidity is neglected), where $\Delta\rho$ is the density difference between the lithosphere and the overlying medium (air or water), and $h_{d,0}$ is a constant.

With these assumptions, we can state the force balance equation for a lithosphere element. For simplicity, this is first done in cartesian coordinates for an element of length dx and width dy:

$$\begin{aligned}
\tau_{xz} dx dy - t_L \frac{\partial \tau_{zz}}{\partial x} dx dy &= t_L \frac{\partial \tau_{xx}}{\partial x} dx dy + t_L \frac{\partial \tau_{yx}}{\partial y} dx dy \\
\tau_{yz} dx dy - t_L \frac{\partial \tau_{zz}}{\partial y} dx dy &= t_L \frac{\partial \tau_{xy}}{\partial x} dx dy + t_L \frac{\partial \tau_{yy}}{\partial y} dx dy
\end{aligned}$$

in accordance with Fig. 1, main text. The first term on the left-hand side is due to stresses acting at the base of the lithosphere, the second term is the "downhill force" resulting from dynamic topography, terms on the right-hand side correspond to stresses transmitted from neighboring lithosphere elements.

In spherical coordinates, the corresponding force balance equations are

$$\begin{aligned}
\tau_{r\vartheta} - \frac{t_L}{r_E} \cdot \frac{\partial \tau_{rr}}{\partial \vartheta} &= t_L \cdot \left[\frac{1}{r_E \sin\vartheta} \frac{\partial (\tau_{\vartheta\vartheta} \sin\vartheta)}{\partial \vartheta} + \frac{1}{r_E \sin\vartheta} \frac{\partial \tau_{\vartheta\varphi}}{\partial \varphi} - \frac{\tau_{\varphi\varphi} \cot\vartheta}{r_E} \right] \\
\tau_{r\varphi} - \frac{t_L}{r_E \sin\vartheta} \cdot \frac{\partial \tau_{rr}}{\partial \varphi} &= t_L \cdot \left[\frac{1}{r_E \sin\vartheta} \frac{\partial (\tau_{\vartheta\varphi} \sin\vartheta)}{\partial \vartheta} + \frac{1}{r_E \sin\vartheta} \frac{\partial \tau_{\varphi\varphi}}{\partial \varphi} + \frac{\tau_{\vartheta\varphi} \cot\vartheta}{r_E} \right] \qquad (1)
\end{aligned}$$

These equations can be derived from more general equations of motion as e.g. given by Bird et al. [1]

Since vertical lithosphere displacements and resulting downhill forces have already been considered, we set $\tau_{rr} = 0$ in the following stress-strain relationships, at the vertically displaced height of the lithosphere, otherwise we would count the effect of radial stresses twice. The stress-strain relationships are different for cases (1) and (2) and we treat the two cases separately in the following.

1.2.1 Viscous incompressible rheology - Eq. 4 main text

The three diagonal components of the stress-strain relationship are

$$\begin{aligned} \tau_{\vartheta\vartheta} &= -\delta p + 2\eta_L \dot{\varepsilon}_{\vartheta\vartheta} \\ \tau_{\varphi\varphi} &= -\delta p + 2\eta_L \dot{\varepsilon}_{\varphi\varphi} \\ \tau_{rr} &= -\delta p + 2\eta_L \dot{\varepsilon}_{rr} = 0 \end{aligned}$$

where δp is the anomalous pressure $(\tau_{rr} + \tau_{\vartheta\vartheta} + \tau_{\varphi\varphi})/3$. Inserting the third equation into the first two gives

$$\begin{aligned} \tau_{\vartheta\vartheta} &= 2\eta(\dot{\varepsilon}_{\vartheta\vartheta} - \dot{\varepsilon}_{rr}) \\ \tau_{\varphi\varphi} &= 2\eta(\dot{\varepsilon}_{\varphi\varphi} - \dot{\varepsilon}_{rr}) \end{aligned}$$

Further using incompressibility, the horizontal components of the stress- strain relationship thus become

$$\begin{aligned} \tau_{\vartheta\vartheta} &= 2\eta(2\dot{\varepsilon}_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) \\ \tau_{\varphi\varphi} &= 2\eta(\varepsilon_{\vartheta\vartheta} + 2\dot{\varepsilon}_{\varphi\varphi}) \qquad (2) \\ \tau_{\vartheta\varphi} &= 2\eta\dot{\varepsilon}_{\vartheta\varphi} \end{aligned}$$

Eqs. 2 are inserted in Eq. 1. Strain rates are expressed in terms of displacement rates. Radial displacement rates v_r are set zero, because the lithosphere is at the upper boundary. The resulting equations therefore only contain displacement rates v_ϑ and v_φ and stress tensor components τ_{rr}, $\tau_{r\vartheta}$ and $\tau_{r\varphi}$. The stress tensor components are expanded in terms of spherical harmonics with Eq. 3 in the main text, and displacement rates are expanded in analogy. Thus Eqs. 2 are written separately for toroidal and poloidal part, and for each spherical harmonic coefficient. After some further transformations, especially making use of the relation

$$\frac{\partial^2 Y_{lm}}{\partial \vartheta^2} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 Y_{lm}}{\partial \varphi^2} + \frac{\partial Y_{lm}}{\partial \vartheta} \cot \vartheta = -l \cdot (l+1) Y_{lm}, \qquad (3)$$

Eq. 4 of the main text is obtained.

1.2.2 Elastic rheology - Eq. 5 of main text)

The three diagonal components of the stress-strain relationship are

$$\begin{aligned}
\tau_{\vartheta\vartheta} &= \left(\kappa_L - \frac{2}{3}\mu_L\right) \cdot (\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) + 2\mu_L \varepsilon_{\vartheta\vartheta} \\
\tau_{\varphi\varphi} &= \left(\kappa_L - \frac{2}{3}\mu_L\right) \cdot (\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) + 2\mu_L \varepsilon_{\varphi\varphi} \\
\tau_{rr} &= \left(\kappa_L - \frac{2}{3}\mu_L\right) \cdot (\varepsilon_{rr} + \varepsilon_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) + 2\mu_L \varepsilon_{rr} = 0.
\end{aligned}$$

We solve the third equation for ε_{rr} and insert it into the first two equations and obtain for the horizontal components of the stress-strain relationship

$$\begin{aligned}
\tau_{\vartheta\vartheta} &= \left(\kappa_L - \frac{2}{3}\mu_L\right) \left(1 - \frac{\kappa - \frac{2}{3}\mu}{\kappa + \frac{4}{3}\mu}\right) (\varepsilon_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) + 2\mu\varepsilon_{\vartheta\vartheta} \\
\tau_{\varphi\varphi} &= \left(\kappa_L - \frac{2}{3}\mu_L\right) \left(1 - \frac{\kappa - \frac{2}{3}\mu}{\kappa + \frac{4}{3}\mu}\right) (\varepsilon_{\vartheta\vartheta} + \varepsilon_{\varphi\varphi}) + 2\mu\varepsilon_{\varphi\varphi} \qquad (4) \\
\tau_{\vartheta\varphi} &= 2\mu\varepsilon_{\vartheta\varphi}
\end{aligned}$$

Eqs. 4 are inserted in Eq. 1. Strains are expressed in terms of displacements. Radial displacements u_r are set zero, because the lithosphere is at the upper boundary. The resulting equations therefore only contain displacements u_ϑ and u_φ and stress tensor components τ_{rr}, $\tau_{r\vartheta}$ and $\tau_{r\varphi}$. The stress tensor components are expanded in terms of spherical harmonics with Eq. 3 in the main text, and displacements are expanded in analogy. Thus Eqs. 4 are written separately for toroidal and poloidal part, and for each spherical harmonic coefficient. Again, after some transformations especially making use of Eq. 3, Eq. 5 of the main text is obtained.

1.2.3 Interpretation

From the equations we can conclude that the lithosphere "integrates" tangential stresses acting at its base: For the sake of the argument, suppose that tangential stresses acting at the base of the lithosphere have a "white" spectrum, i.e. with roughly equal magnitude at each spherical harmonic degree, it follows from Eq. 3 in the main text that $\tau_p \sim 1/l$, hence v_p or $u_p \sim 1/l^3$ according to Eqs. 4 and 5 in the main text, hence lithospheric displacements $\sim 1/l^2$ and strains and stresses $\tau_{\vartheta\vartheta}$, $\tau_{\vartheta\varphi}$ and $\tau_{\varphi\varphi} \sim 1/l$. Therefore the effect of tangential stresses becomes larger with increasing wavelength, in agreement with Fig. 1 in the main text. Obviously this "integrative" effect remains valid for any tangential stress spectrum, and no assumptions of its shape are made in the modelling.

By analogy, one can estimate that a given spectrum of normal stresses acting at the base of the lithosphere causes a spectrum of similar shape for stresses $\tau_{\vartheta\vartheta}$, $\tau_{\vartheta\varphi}$ and $\tau_{\varphi\varphi}$.

2. Details of the calculation of stresses due to topography other than dynamic topography

`http://www.geophysik.uni-frankfurt.de/~steinber/papers/epsl5746_bg2.html`

3. Details of the calculation of "free" plate motion

The torques $\mathbf{T}_i$ in Eq. 6 of the main text linearly depend on the rotation vectors $\boldsymbol{\omega}_j$ of all n plates. The condition that the torque on each plate vanishes can therefore be written in the form

$$\mathbf{T}_i = \mathbf{T}_{i,0} + \sum_{j=1}^{n} \mathbf{A}_{ij}\boldsymbol{\omega}_j = 0 \text{ for } i = 1, \ldots, n \tag{5}$$

The tensors $\mathbf{A_{ij}}$ only depend on geometry and radial viscosity structure and can be computed. $\mathbf{T}_{i,0}$ (the torque acting on each plate for zero plate motions) can also be computed. This set of $3n$ equations is however not linearly independent, as there is no net torque acting on the entire lithosphere, therefore

$$\sum_{i=1}^{n} \mathbf{T}_{i,0} = 0 \text{ and } \sum_{i=1}^{n} \mathbf{A}_{ij} = 0 \text{ for } j = 1, \ldots, n$$

We therefore only use $3n - 3$ of the Eqs. 5, plus the additional condition that the net rotation of the entire lithosphere vanishes, i.e. the integral of $\boldsymbol{\omega}$ over the entire lithosphere vanishes. This condition can be written in the form

$$\int \boldsymbol{\omega} dA = \sum_{i=1}^{n} b_i \boldsymbol{\omega}_i = 0 \tag{6}$$

because the integral linearly depends on the rotation vector of each plate. b_i only depend on plate geometry and can be computed. The three equations 6 together with $3n - 3$ of the Eqs. 5 form a linear system of $3n$ linearly independent equations, which is solved for the $3n$ coefficients of the rotation vectors $\boldsymbol{\omega}_i, i = 1, \ldots, n$.

Dynamic topography h_d and hence other topography h_{sc} slightly depends on plate motions. Therefore, free plate motions that also account for torques due to other topography h_{sc} are determined iteratively: In the first step, $\tau'_{rr,sc}$ corresponding to "other" topography h_{sc} is not included in Eqn. 6 of the main text. A first solution for free plate motions, dynamic topography h_d and "other" topography h_{sc} is computed. In the second step $\tau'_{rr,sc}$ is then included.

4. More detailed comparison of predicted scalar stress magnitudes with observations

In Table 1 we compare our predicted scalar stress magnitudes with observed stress regimes as compiled by Zoback [2]. The agreement is not really good at first sight, however most of the discrepancy can be qualitatively explained by two effects already mentioned in the main text: (1) the relative magnitudes of stress anomalies due to mantle flow tend to be overpredicted, and (2) the compensation depth of topography isostatically compensated below the crust is probably assumed too small in many regions on the continents. For the Tibetan Plateau and the High Andes, the most tensile scalar stress anomalies due to topography isostatically compensated within the crust plus due to ocean floor cooling are in fact predicted. Because of (1), the total predicted scalar stress anomaly is still slightly compressive in the High Andes, whereas observations indicate a tensile stress anomaly. In the oceans, the predicted stress anomalies due to mantle flow tend to be more tensile than on the continents, whereas it is the other way round for the predicted stress anomalies due to other causes. Again because of (1), the predicted stresses in the Pacific Ocean are more tensile than observed, however the "trends" (from more tensile in the Eastern Pacific/Western Indian ocean to more compressive in the Western Pacific/Central Indian Ocean are predicted correctly. In East Asia, the predicted topography isostatically compensated at subcrustal levels tends to be positive, whereas in Africa it tends to be negative. Because of (2), the predicted stresses in East Asia are probably too compressive, whereas in Africa, they are probably too extensive. Again, the trend from most tensile stress anomalies in the East African Rift to most compressive in North Africa is correctly predicted in Africa. On Antarctica, the West Antarctic rift is the only region where no compression is predicted, i.e. the trend is again predicted correctly. Comparison of the two models shows that both predictions are wrong in some places, but often where one of the predictions is wrong, the other one is right. This is also true for the stress directions. It appears that with a more suitable weighting of mantle flow and lithospheric contributions an even better agreement of predicted and observed stress anomaly could be achieved, but this was not attempted here.

5. Results for other tomographic models

`http://www.geophysik.uni-frankfurt.de/~steinber/papers/epsl5746_bg2.html`

References

[1] Bird, Armstrong, Kassager, Dynamics of Polymeric Liquids, Vol. I: Fluid Mechanics, Wiley, 1986, 2nd Ed.

[2] M.L. Zoback, First and second-order patterns of stress in the lithosphere: The World Stress Map Project. J. Geophys. Res. 97 (1992) 11703-11728.

Region	Observation	prediction 1	prediction 2
Midplate North America	T/SS	compressive	$\sim$ neutral to compressive
Continental South America	T/SS	compressive	$\sim$ neutral
High Andes	NF	compressive	tensile
Western Europe	SS	both	$\sim$ neutral to compressive
China/Eastern Asia	SS	compressive	both
Tibetan Plateau	NF	tensile	tensile
East African Rift	NF	tensile	tensile
Western and Southern Africa	SS	tensile	tensile
North Africa	T/SS	both	both
India	T/SS	compressive	both
Central Indian Ocean	T/SS	both	both
West Indian Ocean	NF	tensile	both
Australia	TF	compressive	neutral to compressive
Young Pacific Plate	SS	tensile	tensile
Older Pacific plate	T/SS	both	compressive
West Antarctic rift	NF	$\sim$ neutral	both

Table 1: Prediced scalar stress magnitudes and observed stress regimes. Observations are from Zoback [2], table 4; in accordance we use the abbreviations NF = normal faulting stress regime, SS = strike slip faulting stress regime, TF = thrust faulting stress regime; T/SS = combined thrust and strike-slip regimes, predictions from Fig. 6, main text; "both" = predicted stress anomaly is tensile in part of the region and compressive in another part. Prediction 1 is for the "mantle flow" model shown in Fig. 6 of the main text, prediction 2 of the "lithospheric" model shown in the top panel of Fig. 8 of the main text.